Algebra I Final Exam.

Answer the first question and **ANY 4 OF THE REMAINING 8 QUESTIONS** with carefully reasoned and written proofs. This exam is for 3 hours. As general test taking strategy, answer questions that you find easier first and not in order. Each question is worth 20 marks.

I. Justify with either a **proof** or a **counterexample** whether each of the following statements is true or false:
(a) For a vector space V over $\mathbb{C}$, if $T : V \to V$ is a linear transformation that is onto, then it is also one-to-one.
(b) For a finite-dimenaional vector space V over a field F, if W is a subspace of V, then there is another subspace U of V such that $V = W \oplus U$.
(c) If A is a $m \times n$ matrix over $\mathbb{C}$ and the system $AX = B$ has more than one solution, then it has infinitely many solutions.
(d) Suppose that $\phi : G \to H$ is a group homomorphism and $x \in G$ with $\phi(x) = y \in H$. If $order(x) = r$ then $order(y)$ divides r.
(e) Every group of order pq where p and q are distinct primes is cyclic.

II. Suppose that G is a finite group for which each p-Sylow subgroup is normal for each prime p that divides $order(G)$. Prove that G is isomorphic to the product of its Sylow subgroups.

III. Suppose that p is the smallest prime dividing $order(G)$ and that H is a subgroup of G of index p. Prove that H is normal.

IV. Let V and W be finite-dimensional vector spaces over a field F of dimensions n and m respectively. Let $Hom(V, W)$ be the set of all linear transformations from V to W. For $S, T \in Hom(V, W)$ define $S + T \in Hom(V, W)$ by $(S + T)(v) = S(v) + T(v)$ for all $v \in V$ and for $c \in F$ define $cT \in Hom(V, W)$ by $(cT)(v) = cT(v)$ for all $v \in V$.
(a) Show that this makes $Hom(V, W)$ into a vector space over F.
(b) Show that $dim(Hom(V, W)) = mn$. Hint: Use bases of V and W.
(c) If $T \in Hom(V, V)$ show that there is a non-zero polynomial $a_0 + a_1X + a_2X^2 + \cdots a_kX^k$ over F such that $a_0 + a_1T + a_2T^2 + \cdots a_kT^k = 0$.

V. Let U be the group of all upper triangular 3×3 matrices with all diagonal entries 1. Compute the centre Z of the group U. Establish an isomorphism from the quotient U/Z to $\mathbb{R} \oplus \mathbb{R}$.

VI. Consider the dihedral group D_{10} with two generators R and F satisfying $R^5 = I = F^2$ and $RF = FR^4$. (a) List all its elements and for each, say what its order is.
(b) List all subgroups of D_{10} and prove that any subgroup does occur on your list.
(c) Partition D_{10} into conjugacy classes.
(d) For each element of D_{10} determine its normaliser.

VII. Let A be an $m \times n$ and B be an $p \times n$ matrix over a field F. Suppose that the columns of B^t span the solution space of the system $AX = 0$. Show that the columns of A^t span the solution space of the system $BX = 0$.

VIII. In the vector space $V = \mathbb{C}^5$ consider the subspaces W_1 spanned by $\{(5,7,3,0,9),(2,3,1,2,6)\}$ and W_2 spanned by $\{(1,6,9,2,0),(1,1,4,1,-1),(1,-3,3,1,-3)\}$. Find bases for each of these spaces and for W_1+W_2 and $W_1 \cap W_2$. Is $V = W_1 \oplus W_2$?

IX. Let G be a finite group and consider the group $G \times G$. Let $H = \{(g,g) : g \in G\}$ which is a subgroup of $G \times G$. Let X be a subset of G containing one element from each conjugacy class of G. Prove that the distinct double cosets of H in G are $H(1,x)H$ as x varies over X.